JOURNAL OF APPROXIMATION THEORY 4, 387–400 (1971)

# On Absolute Convergence of Jacobi Series

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*Communicated by Oved Shisha*

Received October 22, 1969

## 1. INTRODUCTION

This paper answers a question concerning the expansion of functions in an absolutely convergent series of Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$(1-x)^\alpha(1+x)^\beta \qquad (\alpha > -1, \beta > -1).$$

They satisfy the relation

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\} \tag{1.1}$$

(see Szegö [5, Section 4.3]), usually called Rodrigues's formula. The orthogonality property is given by

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx = h_n(\alpha,\beta)\, \delta_{m,n} \tag{1.2}$$

with

$$h_n(\alpha,\beta) = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}, \tag{1.3}$$

$\delta_{m,n} = 0$ if $m \neq n$ and $\delta_{m,n} = 1$ if $m = n$.

With a function $f(x)$ we can associate a series:

$$f(x) \sim \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x), \tag{1.4}$$

where

$$a_k = (h_k(\alpha,\beta))^{-1} \int_{-1}^{1} f(x)\, P_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx, \tag{1.5}$$

provided that the integral in (1.5) exists for all $k$. The coefficients $a_k$ are then called the Fourier coefficients of $f(x)$.

DEFINITION. *A function $f(x)$ is said to be in the class $A(\alpha, \beta)$ if $\sum_{k=0}^{\infty} | a_k | \, | P_k^{(\alpha,\beta)}(x)|$ converges uniformly on the interval $-1 \leqslant x \leqslant 1$, where $a_k$ are the Fourier coefficients of $f(x)$.*

It is a well-known fact (see Szegö [5, Section 7.32]), that the Jacobi polynomials reach the maximum of their absolute value on the interval $[-1, 1]$ at $x = 1$, provided that $\alpha \geqslant \beta$ and $\alpha \geqslant -\frac{1}{2}$. Since

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k + \alpha + 1)}{k! \, \Gamma(\alpha + 1)} = O(k^\alpha),$$

it follows that a necessary and sufficient condition for $f(x)$ to be in $A(\alpha, \beta)$ $(\alpha \geqslant \beta, \alpha \geqslant -\frac{1}{2})$ is

$$\sum_{k=0}^{\infty} | a_k | \, k^\alpha < \infty. \tag{1.6}$$

We shall study the question: for which values of $\gamma$ and $\delta$ does

$$f(x) \in A(\alpha, \beta) \qquad \text{imply} \quad f(x) \in A(\gamma, \delta); \tag{A}$$

where $\alpha \geqslant \beta$ and $\alpha \geqslant -\frac{1}{2}$?

In the following it will always be assumed that $\alpha \geqslant \max(\beta, -\frac{1}{2})$, $\beta > -1$.

## 2. THEOREMS

There is a unique way of expressing the polynomials $P_k^{(\alpha,\beta)}(x)$ in terms of the polynomials $P_j^{(\gamma,\delta)}(x)$, $j = 0, 1, 2, \ldots, k$:

$$P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} c_{jk}(\alpha, \beta; \gamma, \delta) \, P_j^{(\gamma,\delta)}(x). \tag{2.1}$$

The coefficients $c_{jk}(\alpha, \beta; \gamma, \delta)$ are defined to be 0 if $j > k$. Rivlin and Wilson [4] have proved the following:

THEOREM 1. *If $\gamma \geqslant \delta$, $\gamma \geqslant -\frac{1}{2}$ and $c_{jk}(\alpha, \beta; \gamma, \delta) \geqslant 0$ for all $j$ and $k$, then relation* (A) *holds.*

*Proof.* Let $f(x) \in A(\alpha, \beta)$. Then

$$\sum_{k=0}^{\infty} | a_k | \, P_k^{(\alpha,\beta)}(1) < \infty,$$

where the $a_k$ are given by (1.5). We now consider the expansion

$$f(x) \sim \sum_{j=0}^{\infty} b_j P_j^{(\gamma,\delta)}(x).$$

Then

$$\begin{aligned} b_j &= (h_j(\gamma, \delta))^{-1} \int_{-1}^{1} f(x)\, P_j^{(\gamma,\delta)}(x)(1-x)^\gamma(1+x)^\delta\, dx \\ &= (h_j(\gamma, \delta))^{-1} \int_{-1}^{1} \left\{ \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x) \right\} P_j^{(\gamma,\delta)}(x)(1-x)^\gamma(1+x)^\delta\, dx \\ &= \sum_{k=0}^{\infty} a_k \left\{ (h_j(\gamma, \delta))^{-1} \int_{-1}^{1} P_k^{(\alpha,\beta)}(x)\, P_j^{(\gamma,\delta)}(x)(1-x)^\gamma(1+x)^\delta\, dx \right\} \\ &= \sum_{k=j}^{\infty} a_k c_{jk}(\alpha, \beta; \gamma, \delta). \end{aligned}$$

The term-by-term integration is justified by the uniform convergence. Since $\gamma \geqslant \delta$ and $\gamma \geqslant -\frac{1}{2}$, we know that

$$\max_{-1 \leqslant x \leqslant 1} | P_j^{(\gamma,\delta)}(x)| = P_j^{(\gamma,\delta)}(1), \qquad j = 0, 1, 2, \ldots.$$

Thus it remains to show that the sequence

$$F_m = \sum_{j=0}^{m} | b_j |\, P_j^{(\gamma,\delta)}(1)$$

is bounded.

Using the fact that $c_{jk}(\alpha, \beta; \gamma, \delta) \geqslant 0$ for all $j$ and $k$, we obtain

$$\begin{aligned} F_m &= \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \left| \sum_{k=j}^{\infty} a_k c_{jk}(\alpha, \beta; \gamma, \delta) \right| \\ &\leqslant \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} | a_k |\, c_{jk}(\alpha, \beta; \gamma, \delta) \\ &\leqslant \sum_{k=0}^{\infty} | a_k | \sum_{j=0}^{m} c_{jk}(\alpha, \beta; \gamma, \delta)\, P_j^{(\gamma,\delta)}(1) \\ &\leqslant \sum_{k=0}^{\infty} | a_k |\, P_k^{(\alpha,\beta)}(1) < \infty. \end{aligned}$$

Q.E.D.

It is known (see Askey [1]) that the positivity condition for $c_{jk}(\alpha, \beta; \gamma, \delta)$ is satisfied in the following cases (see Fig. 1):

(i) $\beta = \delta$ and $\alpha > \gamma$, $\gamma \geqslant \delta$,

(ii) $\alpha = \beta$, $\gamma = \delta$, and $\alpha > \gamma$,

(iii) $\alpha = \gamma$, $\beta = \delta - n$ ($n$ a positive integer), $\gamma \geqslant \delta$.

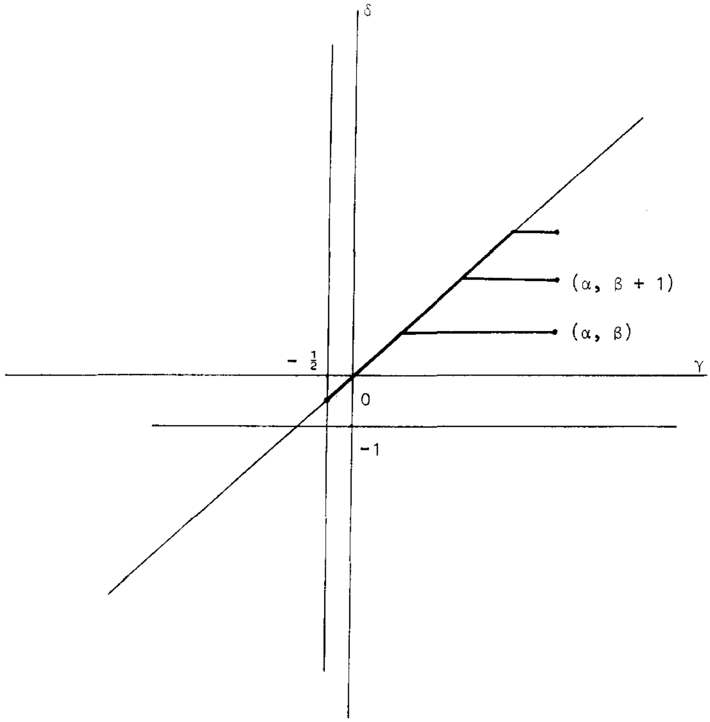


FIGURE 1.

We shall prove now, that relation (A) holds in the following cases:

(i) $\alpha = \gamma$, $\beta < \delta$, $\gamma \geqslant \delta$,

(ii) $\alpha = \gamma + \mu$, $\beta = \delta + \mu$, $\mu > 0$, $\gamma \geqslant \max(\delta, -\frac{1}{2})$, $\delta > -1$.

THEOREM 2. *If* $\gamma = \alpha$ *and* $\delta = \beta + \mu$, *where* $\mu > 0$ *and* $\gamma \geqslant \delta$, *then relation* (A) *holds.*

*Proof.* Following the proof of Theorem 1, it remains to show that the sequence

$$F_m = \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \left| \sum_{k=j}^{\infty} a_k c_{jk}(\alpha, \beta; \gamma, \delta) \right|$$

is bounded.

We now have

$$F_m \leqslant \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} | a_k | \, | c_{jk}(\alpha, \beta; \gamma, \delta)|$$

$$\leqslant \sum_{k=0}^{\infty} | a_k | \sum_{j=0}^{m} | c_{jk}(\alpha, \beta; \gamma, \delta)| \, P_j^{(\gamma,\delta)}(1).$$

As

$$P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} c_{jk}(\alpha, \beta; \alpha, \beta + \mu) \, P_j^{(\alpha,\beta+\mu)}(x),$$

it follows from the identity

$$P_k^{(\alpha,\beta)}(x) = (-1)^n \, P_n^{(\beta,\alpha)}(-x) \qquad \text{(see Szegö [5, Section 4.1])}$$

that

$$P_k^{(\beta,\alpha)}(x) = \sum_{j=0}^{k} (-1)^{k-j} c_{jk}(\alpha, \beta; \alpha, \beta + \mu) \, P_j^{(\beta+\mu,\alpha)}(x).$$

In Section 9.4 of Szegö [5] the following relation is derived:

$$P_k^{(\beta,\alpha)}(x) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(-\mu)\, \Gamma(k + \alpha + \beta + 1)} \times \sum_{j=0}^{k} \frac{\left\{ \begin{array}{l} \Gamma(k + j + \alpha + \beta + 1)\, \Gamma(k - j - \mu) \\ \quad \times \Gamma(j + \alpha + \beta + \mu + 1)(2j + \alpha + \beta + \mu + 1) \end{array} \right\}}{\Gamma(k + j + \alpha + \beta + \mu + 2)\, \Gamma(k - j + 1)\, \Gamma(j + \alpha + 1)} \times P_j^{(\beta+\mu,\alpha)}(x).$$

Hence

$$F_m \leqslant \sum_{k=j}^{\infty} | a_k | \sum_{j=0}^{k} \left| \frac{\left\{ \begin{array}{l} \Gamma(k + \alpha + 1)\, \Gamma(k + j + \alpha + \beta + 1)\, \Gamma(k - j - \mu) \\ \quad \times \Gamma(j + \alpha + \beta + \mu + 1)(2j + \alpha + \beta + \mu + 1) \end{array} \right\}}{\left\{ \begin{array}{l} \Gamma(-\mu)\, \Gamma(k + \alpha + \beta + 1)\, \Gamma(k + j + \alpha + \beta + \mu + 2) \\ \quad \times \Gamma(k - j + 1)\, \Gamma(j + \alpha + 1) \end{array} \right\}} \right| \times P_j^{(\alpha,\beta+\mu)}(1).$$

Since $\Gamma(k + \alpha)/\Gamma(k) = O(k^\alpha)$, we can estimate the order of magnitude of $F_m$.

$$F_m \leqslant c \sum_{k=0}^{\infty} | a_k | \, k^{-\beta} \sum_{j=0}^{k} (k + j)^{-\mu-1}(k - j)^{-\mu-1} j^{\alpha+\beta+\mu+1}$$

$$\leqslant c \sum_{k=0}^{\infty} | a_k | \, k^{-\beta-\mu-1} \left( \sum_{j=0}^{[k/2]} k^{-\mu-1} j^{\alpha+\beta+\mu+1} + \sum_{j=[k/2]+1}^{k} k^{\alpha+\beta+\mu+1}(k - j)^{-\mu-1} \right)$$

$$\leqslant c \sum_{k=0}^{\infty} | a_k | \, k^\alpha < \infty.$$

THEOREM 3. *If* $\gamma = \alpha - \mu$ *and* $\delta = \beta - \mu$, *where* $\mu > 0$ *and* $\gamma \geqslant \max(\delta, -\frac{1}{2})$, $\delta > -1$, *then relation* (A) *holds.*

*Proof.* It suffices to show that

$$\sum_{j=0}^{k} | c_{jk}(\alpha, \beta; \alpha - \mu, \beta - \mu)| \, P_j^{(\alpha-\mu,\beta-\mu)}(1) = O(k^\alpha).$$

Substituting the values of $c_{jk}(\alpha, \beta; \alpha - \mu, \beta - \mu)$, we obtain

$$\sum_{j=0}^{k} P_j^{(\alpha-\mu,\beta-\mu)}(1)(h_j(\alpha - \mu, \beta - \mu))^{-1}$$
$$\times \left| \int_{-1}^{1} P_k^{(\alpha,\beta)}(x) \, P_j^{(\alpha-\mu,\beta-\mu)}(x)(1 - x)^{\alpha-\mu}(1 + x)^{\beta-\mu} \, dx \right|$$
$$= \left( \sum_{j=0}^{k} \frac{\Gamma(j + \alpha + \beta - 2\mu + 1)(2j + \alpha + \beta - 2\mu + 1)}{\Gamma(\alpha - \mu + 1) \, \Gamma(j + \beta - \mu + 1)} \right)$$
$$\times \left| \int_0^\pi P_k^{(\alpha,\beta)}(\cos\theta) \, P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{2\alpha-2\mu+1} \left(\cos\frac{\theta}{2}\right)^{2\beta-2\mu+1} d\theta \right|.$$

We will take the liberty of omitting lower order terms in $k$ when they are inessential.

We shall take the integral over $[0, \pi/2]$ only. The interval $[\pi/2, \pi]$ can be handled similarly. It suffices to show that

$$\left( \sum_{j=0}^{k} j^{\alpha-\mu+1} \right)$$
$$\times \left| \int_0^{\pi/2} \left(\sin\frac{\theta}{2}\right)^{2\alpha-2\mu+1} \left(\cos\frac{\theta}{2}\right)^{2\beta-2\mu+1} P_k^{(\alpha,\beta)}(\cos\theta) \, P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta) \, d\theta \right|$$
$$= O(k^\alpha).$$

We need the following estimates for Jacobi polynomials and Bessel functions:

$$| P_n^{(\alpha,\beta)}(\cos\theta)| \leqslant An^\alpha, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \tag{2.2}$$

(Szegö [5, 7.32.6]),

$$| P_n^{(\alpha,\beta)}(\cos\theta)| \leqslant An^{-1/2}\theta^{-\alpha-1/2}, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \tag{2.3}$$

$$| J_\alpha(x)| \leqslant Ax^\alpha, \qquad 0 \leqslant x \leqslant 1, \qquad \text{(Szegö [5, 1.71.10])}, \tag{2.4}$$

$$| J_\alpha(x)| \leqslant Ax^{-1/2}, \qquad x \geqslant 1, \qquad \text{(Szegö [5, 1.71.11])}, \tag{2.5}$$

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), \qquad \text{(Szegö 5, 1.71.7])}. \tag{2.6}$$

We shall also need the Sonine integral

$$\int_0^\infty \frac{J_\mu(at)\,J_\nu(bt)}{b^\nu t^{\mu-\nu-1}}\,dt = \frac{(a^2-b^2)^{\mu-\nu-1}}{2^{\mu-\nu-1}a^\mu\Gamma(\mu-\nu)}, \quad a > b \text{ (Watson [6, Section 13.46])} \tag{2.7}$$

and Hilb's formula

$$\left(\sin\frac{\theta}{2}\right)^\alpha \left(\cos\frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_\alpha(N\theta) + \begin{cases}\theta^{1/2}0(n^{-3/2}), & \text{if } cn^{-1} \leqslant \theta \leqslant \pi-\epsilon,\\ \theta^{\alpha+2}0(n^\alpha), & \text{if } 0 < \theta < cn^{-1},\end{cases}$$

where $N = n + (\alpha+\beta+1)/2$; $c$ and $\epsilon$ are fixed positive numbers [5, 8.21.17].

We follow the method used by Askey and Wainger [2], and therefore wish to replace

$$2^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-\mu+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-\mu+1/2} P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)$$

by $\theta^{1/2}J_{\alpha-\mu}(J\theta)$, $J = j + (\alpha+\beta-2\mu+1)/2$, using Hilb's formula (2.8).

We have then to consider

$$\begin{aligned} I = \sum_{j=0}^{k} j^{\alpha-\mu+1}\Bigg|\int_0^{\pi/2}&\left(\sin\frac{\theta}{2}\right)^{\alpha-\mu+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-\mu+1/2} P_k^{(\alpha,\beta)}(\cos\theta)\\ &\times\Bigg\{2^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-\mu+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-\mu+1/2} P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)\\ &-\frac{J^{-\alpha+\mu}\Gamma(j+\alpha-\mu+1)}{\Gamma(j+1)}\,\theta^{1/2}J_{\alpha-\mu}(J\theta)\Bigg\}\,d\theta\Bigg|. \end{aligned}$$

Setting $I = I_1 + I_2$, where, in $I_1$, the range of integration is $[1/k, \pi/2]$ and in $I_2$, $[0, 1/k]$, and using some of the estimates mentioned above, we get

$$\begin{aligned} I_1 &= O\left(\sum_{j=0}^{k} j^{\alpha-\mu+1}\int_{1/k}^{\pi/2} k^{-1/2}\theta^{-\alpha-1/2}\theta j^{-3/2}\theta^{\alpha-\mu+1/2}\,d\theta\right)\\ &= O\left(k^{\alpha-\mu}\int_{1/k}^{\pi/2}\theta^{1-\mu}\,d\theta\right)\\ &= O(k^{\alpha-\mu}(c + k^{\mu-2} + \delta_{\mu,2}\log k))\\ &= O(k^\alpha).\\ I_2 &= O\left(\sum_{j=0}^{k} j^{\alpha-\mu+1}\int_0^{1/k} k^\alpha\theta k^{-3/2}\theta^{\alpha-\mu+1/2}\,d\theta\right)\\ &= O\left(k^{2\alpha-\mu+1/2}\int_0^{1/k}\theta^{\alpha-\mu+3/2}\,d\theta\right)\\ &= O(k^{\alpha-2}). \end{aligned}$$

The process of replacing the other Jacobi polynomial by the appropriate Bessel function is similar.

Thus we are led to investigate

$$L = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\pi/2} \left(\sin\frac{\theta}{2}\right)^{-\mu} \left(\cos\frac{\theta}{2}\right)^{-\mu} \theta J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right|$$

where $K = k + (\alpha + \beta + 1)/2$. We want to replace $(\sin \theta/2)^{-\mu} (\cos \theta/2)^{-\mu}$ by $\theta^{-\mu}$. It is easily seen that $(\sin \theta/2)^{-\mu} (\cos \theta/2)^{-\mu} = (\theta/2)^{-\mu} G(\theta)$, where $G(0) = 1$, $G(\theta)$ is bounded and $1 - G(\theta) = O(\theta^2)$. Thus we have to consider

$$E = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\pi/2} \theta^{1-\mu}(1 - G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right|.$$

We set $E = E_1 + E_2$, where in $E_1$ the range of integration is $[0, 1/k]$, and in $E_2$, $[1/k, \pi/2]$.

Applying some of the estimates mentioned above, we get

$$\begin{aligned} E_1 &= \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{1/k} \theta^{1-\mu}(1 - G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| \\ &= O\left( \sum_{j=0}^{k} j^{\alpha-\mu+1} j^{\alpha-\mu} k^{\alpha} \int_0^{1/k} \theta^{2\alpha-\mu+3-\mu}\, d\theta \right) \\ &= O(k^{\alpha-2}). \end{aligned}$$

Using the asymptotic formula for Bessel functions and the error term, we obtain, for $\mu < 1$,

$$\begin{aligned} E_2 &= \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{1/k}^{\pi/2} \theta^{1-\mu}(1 - G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| \\ &= O\left( k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \left| \int_{1/k}^{\pi/2} \theta^{-\mu}(1 - G(\theta))\, e^{i(J \pm K)\theta}\, d\theta \right| \right) \\ &\quad + O\left( k^{-3/2} \sum_{j=0}^{k} j^{\alpha-\mu-1/2} \int_{1/k}^{\pi/2} \theta^{-\mu}\, d\theta \right) \\ &= O\left( k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \frac{1}{K \pm J} \right) + O(k^{\alpha-\mu-1} + k^{\alpha-2}) \\ &= O(k^{\alpha-\mu}) + O\left( k^{-1/2} \sum_{j=0}^{[k/2]} \frac{j^{\alpha-\mu+1/2}}{k - j} + k^{-1/2} \sum_{j=[k/2]+1}^{k} \frac{j^{\alpha-\mu+1/2}}{k - j} \right) \\ &= O(k^{\alpha-\mu}) + O(k^{\alpha-\mu}) + O(k^{\alpha-\mu} \log k) \\ &= O(k^{\alpha}). \end{aligned}$$

The case $\mu \geqslant 1$ is easily handled:

$$\begin{aligned}
E_2 &= O\left(\sum_{j=1}^{k} j^{\alpha-\mu+1} \left| \int_{1/k}^{\pi/2} \theta^{3-\mu} j^{-1/2} k^{-1/2} \theta^{-1}\, d\theta \right|\right) \\
&= \begin{cases} O(k^{\alpha-\mu+1}(c + k^{\mu-3})), & \mu \neq 3, \\ O(k^{\alpha-2} \log k), & \mu = 3, \end{cases} \\
&= O(k^{\alpha}).
\end{aligned}$$

Finally, we want to replace the range of integration $[0, \pi/2]$ by $[0, \infty)$. Therefore we investigate

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{\pi/2}^{\infty} \theta^{1-\mu} J_{\alpha-\mu}(J\theta)\, J_{\alpha}(K\theta)\, d\theta \right| = A_1 + A_2$$

by using (2.6). Here $A_1$ contains the main terms and $A_2$ all the error terms.

$$\begin{aligned}
A_1 &= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \left| \int_{\pi/2}^{\infty} \theta^{-\mu} e^{i(K \pm J)\theta}\, d\theta \right|\right) \\
&= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} (k \pm j)^{-1}\right) \\
&= O(k^{\alpha-\mu} \log k).
\end{aligned}$$

$$A_2 = O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu-1/2} \int_{\pi/2}^{\infty} \theta^{-\mu-1}\, d\theta\right) = O(k^{\alpha-\mu}).$$

Up to an error term that we have estimated, we may write for $L$,

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{0}^{\infty} \theta^{1-\mu} J_{\alpha-\mu}(J\theta) J_{\alpha}(K\theta)\, d\theta \right|.$$

Using Sonine's integral (2.7), this leads to

$$\begin{aligned}
&\sum_{j=0}^{k} j^{\alpha-\mu+1} \frac{2^{1-\mu} J^{\alpha-\mu} (K^2 - J^2)^{\mu-1}}{K^{\alpha} \Gamma(\mu)} \\
&\quad = O\left(k^{-\alpha} \sum_{j=0}^{k} j^{2\alpha-2\mu+1} (k + j)^{\mu-1} (k - j)^{\mu-1}\right) \\
&\quad = O\left(k^{-\alpha+\mu-1} \left\{ \sum_{j=0}^{[k/2]} j^{2\alpha-2\mu+1} (k - j)^{\mu-1} + \sum_{j=[k/2]+1}^{k} j^{2\alpha-2\mu+1} (k - j)^{\mu-1} \right\}\right) \\
&\quad = O(k^{\alpha}).
\end{aligned}$$

Combining all the estimates, we have shown that

$$\sum_{j=0}^{k} | c_{jk}(\alpha, \beta; \alpha - \mu, \beta - \mu)| P_j^{(\alpha-\mu,\beta-\mu)}(1) = O(k^\alpha),$$

which proves Theorem 3.

## 3. Results

Combining Theorems 1, 2 and 3, we see that for all $(\gamma, \delta)$ in the shaded region of Fig. 2, relation (A) holds. We shall show now by means of examples that that region is exactly the set of all $(\gamma, \delta)$ with $\gamma \geqslant -\frac{1}{2}$, for which (A) holds.

Consider, first, the function $(1 + x)^\mu$, $\mu > 0$. Its Fourier coefficients are

$$a_n = h_n(\alpha, \beta))^{-1} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)(1 - x)^\alpha(1 + x)^{\beta+\mu}\, dx.$$

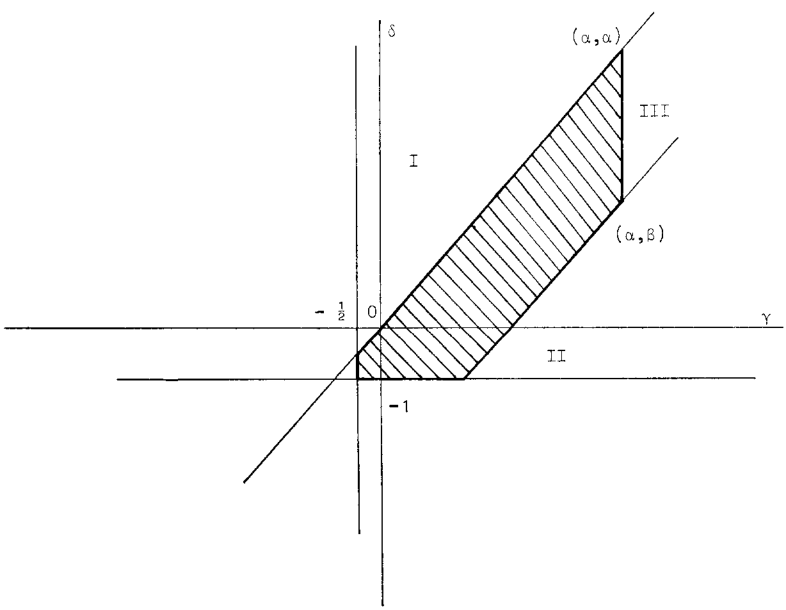


FIGURE 2.

Using Rodrigues's formula (1.1) and integrating by parts, we have

$$a_n = \frac{(-1)^n}{2^n n!\, h_n(\alpha, \beta)} \int_{-1}^{1} (1+x)^\mu \left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}\, dx$$

$$= \frac{\Gamma(\mu+1)}{2^n n!\, h_n(\alpha, \beta)\, \Gamma(\mu - n + 1)} \int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{\beta+\mu}\, dx$$

$$= (-1)^{n+1} \frac{2^\mu}{\pi} \Gamma(\mu+1) \sin \mu\pi\, \Gamma(\beta+\mu+1)(2n+\alpha+\beta+1)$$

$$\times \frac{\Gamma(n+\alpha+\beta+1)\, \Gamma(n-\mu)}{\Gamma(n+\alpha+\beta+\mu+2)\, \Gamma(n+\beta+1)}. \tag{3.1}$$

Thus

$$| a_n | = O(n^{-\beta-2\mu-1}).$$

It follows that $(1+x)^\mu \in A(\alpha, \beta)$ if $\alpha - \beta < 2\mu$.

From (3.1) it is easily derived that the function $(1+x)^\mu$, with $(\alpha-\beta)/2 < \mu < (\gamma - \delta)/2$, $\mu$ not an integer, belongs to $A(\alpha, \beta)$ but not to $A(\gamma, \delta)$. Thus we have found a function for which relation (A) fails in region II of Fig. 2.

In the same way we can calculate the Fourier coefficients of the function $(1-x)^\mu$ and obtain

$$| a_n | = O(n^{-\alpha-2\mu-1}).$$

It follows that $(1-x)^\mu \in A(\alpha, \beta)$ if $\mu > 0$.

But if $\delta > \gamma$, the maximum of the absolute value of the Jacobi polynomials is assumed at $x = -1$ and $P_n^{(\gamma,\delta)}(-1) = O(n^\delta)$. If $\delta > \gamma$, the function $(1-x)^\mu$, with $0 < \mu < (\delta - \gamma)/2$, $\mu$ not an integer, belongs to $A(\alpha, \beta)$ but not to $A(\gamma, \delta)$. Thus, (A) is not valid in region I of Fig. 2.

In order to decide whether relation (A) holds in region III, we study the function $| x |^\mu$. Here

$$a_n = (h_n(\alpha, \beta))^{-1} \int_{-1}^{1} | x |^\mu P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx$$

$$= (h_n(\alpha, \beta))^{-1} \left\{\int_0^1 x^\mu P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx\right.$$

$$\left. + (-1)^n \int_0^1 x^\mu P_n^{(\beta,\alpha)}(x)(1-x)^\beta(1+x)^\alpha\, dx\right\}.$$

If Re $\mu > n - 1$, we can use Rodrigues's formula and integrate by parts. We obtain

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\beta+1)\,\Gamma(\alpha+\mu+2)}$$
$$\times\ {}_2F_1(\mu-n+1, -\beta-n; \alpha+\mu+2; -1)$$
$$+ (-1)^n \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(\beta+\mu+2)}$$
$$\times\ {}_2F_1(\mu-n+1, -\alpha-n; \beta+\mu+2; -1). \tag{3.2}$$

The hypergeometric series ${}_2F_1\ (a, b; c; -1)$ is absolutely convergent if $\mathrm{Re}(a+b-c) < 0$, which means here $-\alpha-\beta-2n-1 < 0$. This is always satisfied (if $n \geqslant 1$). In this case ${}_2F_1(a, b; c; -1)$ is an analytic function of the parameters $a$, $b$ and $c$. Since for Re $\mu > n-1$, $a_n$ is given by (3.2), it follows by analytic continuation that (3.2) holds for all $\mu$ with Re $\mu > -1$. Using the simple relation

$$ {}_2F_1(a, b; c; z) = (1-z)^{-b}\,{}_2F_1\left(b, c-a; c; \frac{z}{z-1}\right)$$
$$= (1-z)^{-b}\,{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right)$$

[3, Section 3.8, (4)], $a_n$ can be written in the following way:

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+1}\Gamma(n+\beta+1)\,\Gamma(\alpha+\mu+2)}$$
$$\times\ {}_2F_1(\alpha+n+1, -\beta-n; \alpha+\mu+2; \tfrac{1}{2})$$
$$+ (-1)^n \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\beta+1}\Gamma(n+\alpha+1)\,\Gamma(\beta+\mu+2)}$$
$$\times\ {}_2F_1(\beta+n+1, -\alpha-n; \beta+\mu+2; \tfrac{1}{2}).$$

An asymptotic expansion of the hypergeometric function in this case, for large $n$, has been given by Watson [7].

The leading term is

$$ {}_2F_1\left(a+n, b-n; c; \frac{1-z}{2}\right) \sim \frac{2^{a+b-1}\Gamma(1-b+n)\,\Gamma(c)(1+e^{-\zeta})^{c-a-b-1/2}}{(n\pi)^{1/2}\Gamma(c-b+n)(1-e^{-\zeta})^{c-1/2}}$$
$$\times \{e^{(n-b)\zeta} + \exp[\pm i\pi(c-\tfrac{1}{2})]\, e^{-(n+a)\zeta}\}$$

where $\zeta$ is defined by $z = \cosh \zeta$ and $\operatorname{Re} \zeta \geqslant 0$, $-\pi \leqslant \operatorname{Im} \zeta \leqslant \pi$. The upper (lower) sign is taken if $\operatorname{Im} z > (<) 0$. In the case in which $z - 1$ is real and negative it is supposed that $z$ attains its value by a limiting process which then determines if $\arg(z - 1)$ is $\pi$ or $-\pi$. The discontinuity in the formula is only apparent; if $z$ crosses the real axis between $\pm 1$, account has to be taken of the discontinuity in the value of $\operatorname{Im} \zeta$. Therefore,

$$| a_n | = O\left(\frac{n^{\alpha+1}\Gamma(n+\beta+1)}{n^{1/2}\Gamma(n+\alpha+\beta+\mu+2)} + \frac{n^{\beta+1}\Gamma(n+\alpha+1)}{n^{1/2}\Gamma(n+\alpha+\beta+\mu+2)}\right)$$
$$= O(n^{-\mu-1/2}). \tag{3.3}$$

Thus, in the case that $\mu > \alpha + \frac{1}{2}$, the function $| x |^{\mu}$ belongs to $A(\alpha, \beta)$.

In the ultraspherical case ($\alpha = \beta$), the Fourier coefficients can easily be calculated. We have

$$a_n = (h_n(\alpha, \alpha))^{-1} \int_{-1}^{1} | x |^{\mu} P_n^{(\alpha,\alpha)}(x)(1 - x^2)^{\alpha}\, dx.$$

Because $| x |^{\mu}$ is an even function, the Fourier coefficients vanish for odd $n$. Application of a well-known formula for ultraspherical polynomials (see Szegö [5, 4.1.5]) yields

$$a_{2n} = \frac{2n!\, \Gamma(2n+\alpha+1)}{h_{2n}(\alpha,\alpha)(2n)!\, \Gamma(n+\alpha+1)} \int_0^1 P_n^{(\alpha,-1/2)}(y)(1-y)^{\alpha}(1+y)^{(\mu-1)/2}\, dy$$
$$= \frac{(-1)^n(4n+2\alpha+1)\, \Gamma(2n+2\alpha+1)\, \Gamma(\mu+1) \sin(\mu/2)\pi\Gamma(n-(\mu/2))}{2^{2\alpha+\mu+1}\Gamma(2n+\alpha+1)\, \Gamma(n+\alpha+(\mu/2)+\frac{3}{2})\, \pi^{1/2}}. \tag{3.4}$$

From (3.3) and (3.4) it follows that if $\gamma > \alpha$, the function $| x |^{\mu}$, with $\alpha + \frac{1}{2} < \mu < \gamma + \frac{1}{2}$, $\mu$ not an even integer, belongs to $A(\alpha, \beta)$ but not to $A(\gamma, \gamma)$. Combined with Theorem 2, this leads to the conclusion that relation (A) cannot hold in region III of Fig. 2.

Thus the shaded region in Fig. 2 is exactly the set (if $\gamma \geqslant -\frac{1}{2}$) where relation (A) holds.

By using the identity $P_n^{(\alpha,\beta)}(x) = (-1)^n P^{(\beta,\alpha)}(-x)$, similar results can be obtained when $\alpha < \beta$.

## Acknowledgment

The author is much indebted to Professor R. A. Askey for setting the problem and for many helpful suggestions for its solution.

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